CTL Model Checking

Lecture #20 of Model Checking

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Overview Lecture #20

⇒ Existential normal form

- Basic CTL model-checking algorithm
- Algorithms for $\exists(\Phi \, \mathsf{U} \, \Psi)$ and $\exists\Box \, \Phi$
- Time complexity

Existential normal form (ENF)

The set of CTL formulas in *existential normal form* (ENF) is given by:

$$\Phi \quad ::= \quad \text{true} \;\Big|\; a \;\Big|\; \Phi_1 \;\wedge\; \Phi_2 \;\Big|\; \neg\Phi \;\Big|\; \exists\bigcirc\,\Phi \;\Big|\; \exists(\Phi_1\,\mathsf{U}\,\Phi_2) \;\Big|\; \exists\Box\,\Phi$$

For each CTL formula, there exists an equivalent CTL formula in ENF

$$\begin{aligned} \forall\bigcirc\,\Phi \quad &\equiv \quad \neg\exists\bigcirc\,\neg\Phi \\ \forall(\Phi\,\mathsf{U}\,\Psi) \quad &\equiv \quad \neg\exists(\neg\Psi\,\mathsf{U}\,(\neg\Phi\wedge\neg\Psi)) \;\wedge\; \neg\exists\Box\,\neg\Psi \end{aligned}$$

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Model checking CTL

- How to check whether *TS* satisfies CTL formula $\widehat{\Phi}$?
  - convert the formula $\widehat{\Phi}$ into the equivalent Φ in ENF
  - compute *recursively* the set $Sat(\Phi) = \{ s \in S \mid s \models \Phi \}$
  - $TS \models \Phi$ if and only if each initial state of *TS* belongs to $Sat(\Phi)$

- Recursive bottom-up computation of $Sat(\Phi)$:
  - consider the parse-tree of Φ
  - start to compute $Sat(a_i)$, for all leafs in the tree
  - then go one level up in the tree and determine $Sat(\cdot)$ for these nodes

$$\text{e.g.,:}\quad Sat(\underbrace{\Psi_1 \wedge \Psi_2}_{\text{node at level } i}) = Sat(\underbrace{\Psi_1}_{\substack{\text{node at}\\ \text{level } i+1}}) \cap Sat(\underbrace{\Psi_2}_{\substack{\text{node at}\\ \text{level } i+1}})$$

  - then go one level up and determine $Sat(\cdot)$ of these nodes
  - and so on....... until the root is treated, i.e., $Sat(\Phi)$ is computed

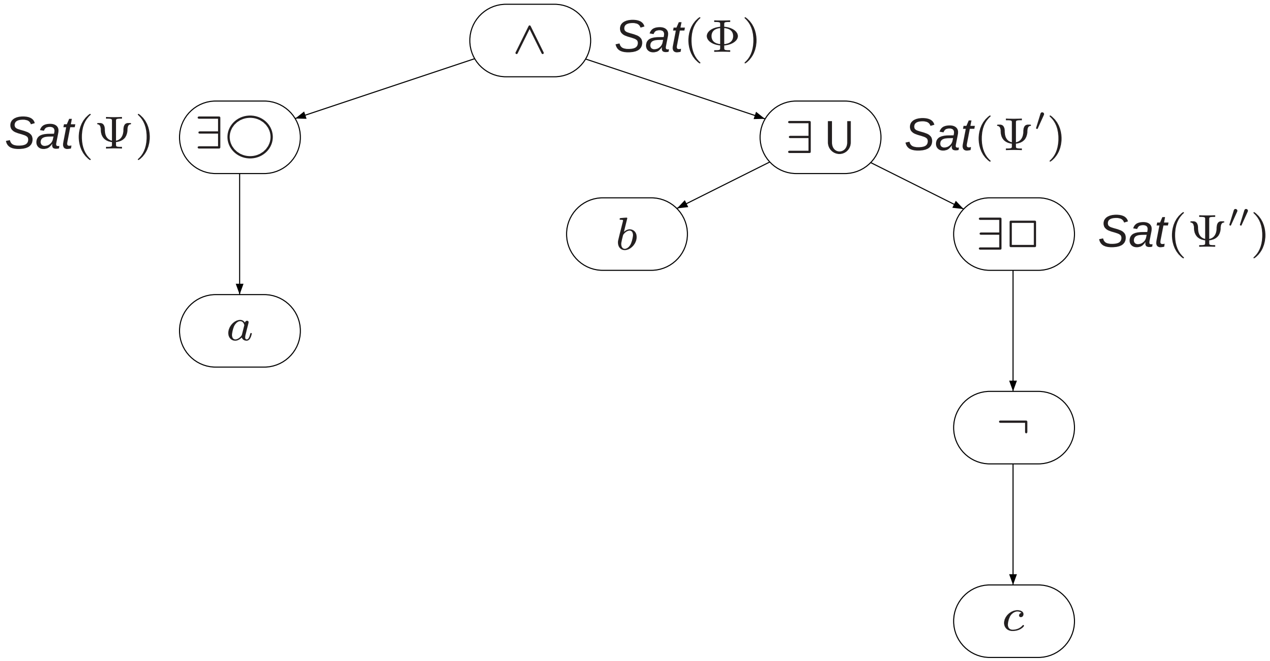
Basic algorithm

Input: finite transition system *TS* and CTL formula Φ (both over *AP*)
Output: *TS* $\models \Phi$

(* compute the sets *Sat*$(\Phi) = \{ s \in S \mid s \models \Phi \}$ *)

for all $i \leqslant |\,\Phi\,|$ **do**
  for all $\Psi \in$ *Sub*(Φ) with $|\,\Psi\,| = i$ **do**
    compute *Sat*(Ψ) from *Sat*(Ψ') (* for maximal proper $\Psi' \in$ *Sub*(Ψ) *)
  od
od
return $I \subseteq$ *Sat*(Φ)

Example



$$\Phi = \underbrace{\exists \bigcirc a}_{\Psi} \wedge \underbrace{\exists(b \, \mathsf{U} \, \underbrace{\exists \Box \neg c}_{\Psi''})}_{\Psi'} \quad .$$

Characterization of *Sat* (1)

For all CTL formulas Φ, Ψ over *AP* it holds:

$$
\begin{aligned}
\mathit{Sat}(\text{true}) &= S \\
\mathit{Sat}(a) &= \{\, s \in S \mid a \in L(s) \,\}, \text{ for any } a \in AP \\
\mathit{Sat}(\Phi \wedge \Psi) &= \mathit{Sat}(\Phi) \cap \mathit{Sat}(\Psi) \\
\mathit{Sat}(\neg \Phi) &= S \setminus \mathit{Sat}(\Phi) \\
\mathit{Sat}(\exists \bigcirc \Phi) &= \{\, s \in S \mid \mathit{Post}(s) \cap \mathit{Sat}(\Phi) \neq \varnothing \,\}
\end{aligned}
$$

where $TS = (S, \mathit{Act}, \rightarrow, I, AP, L)$ is a transition system without terminal states

Characterization of *Sat* (2)

- $\mathit{Sat}(\exists(\Phi \, \mathsf{U} \, \Psi))$ is the <u>smallest</u> subset T of S, such that:

$$(1)\ \mathit{Sat}(\Psi) \subseteq T \quad \text{and} \quad (2)\ \left(s \in \mathit{Sat}(\Phi) \text{ and } \mathit{Post}(s) \cap T \neq \varnothing\right) \;\Rightarrow\; s \in T$$

- $\mathit{Sat}(\exists\Box\,\Phi)$ is the <u>largest</u> subset T of S, such that:

$$(3)\ T \subseteq \mathit{Sat}(\Phi) \quad \text{and} \quad (4)\ s \in T \text{ implies } \mathit{Post}(s) \cap T \neq \varnothing$$

where $\mathit{TS} = (S, \mathit{Act}, \rightarrow, I, \mathit{AP}, L)$ is a transition system without terminal states

Proof

Computation of *Sat*

switch(Φ):

```
a            :  return { s ∈ S | a ∈ L(s) };
...          :  ......
∃◯ Ψ         :  return { s ∈ S | Post(s) ∩ Sat(Ψ) ≠ ∅ };
∃(Φ1 U Φ2)   :  T := Sat(Φ2);     (* compute the smallest fixed point *)
                while { s ∈ Sat(Φ1) \ T | Post(s) ∩ T ≠ ∅ } ≠ ∅ do
                  let s ∈ { s ∈ Sat(Φ1) \ T | Post(s) ∩ T ≠ ∅ };
                  T := T ∪ { s };
                od;
                return T;
∃□ Φ         :  T := Sat(Φ);      (* compute the greatest fixed point *)
                while { s ∈ T | Post(s) ∩ T = ∅ } ≠ ∅ do
                  let s ∈ { s ∈ T | Post(s) ∩ T = ∅ };
                  T := T \ { s };
                od;
                return T;
```

end switch

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$\Rightarrow$ Algorithms for $\exists(\Phi \, \mathsf{U} \, \Psi)$ and $\exists\Box\,\Phi$

- Time complexity

Computing $Sat(\exists(\Phi \mathsf{U} \Psi))$ (1)

- $Sat(\exists(\Phi \mathsf{U} \Psi))$ is the smallest set $T \subseteq S$ such that:

$$(1)\ Sat(\Psi) \subseteq T \quad \text{and} \quad (2)\ (s \in Sat(\Phi) \text{ and } Post(s) \cap T \neq \varnothing) \ \Rightarrow \ s \in T$$

- This suggests to compute $Sat(\exists(\Phi \mathsf{U} \Psi))$ iteratively:

$$T_0 = Sat(\Psi) \quad \text{and} \quad T_{i+1} = T_i \cup \{ s \in Sat(\Phi) \mid Post(s) \cap T_i \neq \varnothing \}$$

- T_i = states that can reach a Ψ-state in at most i steps via a Φ-path
- By induction on j it follows:

$$T_0 \subseteq T_1 \subseteq \ldots \subseteq T_j \subseteq T_{j+1} \subseteq \ \ldots \ \subseteq \ Sat(\exists(\Phi \mathsf{U} \Psi))$$

Computing *Sat*$(\exists(\Phi \,\mathsf{U}\, \Psi))$ (2)

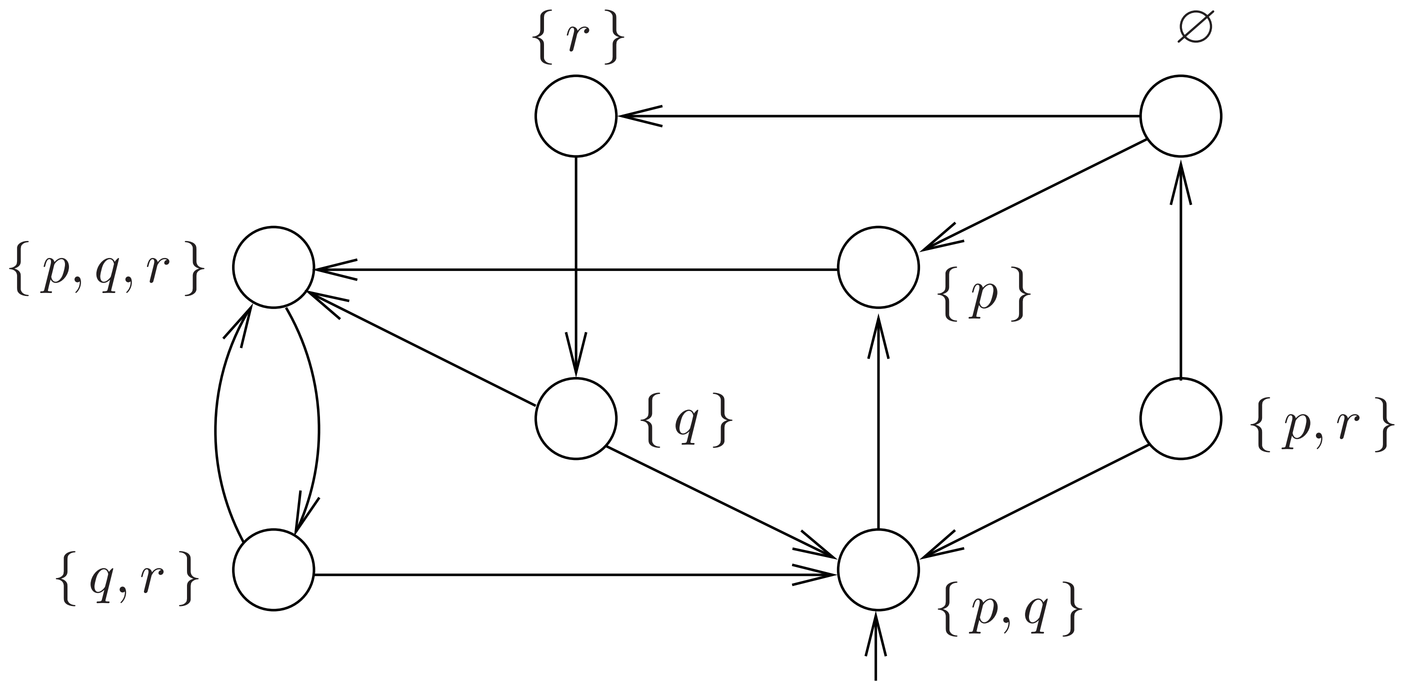
- *TS* is finite, so for some $j \geqslant 0$ we have: $T_j = T_{j+1} = T_{j+2} = \ldots$
- Therefore: $T_j = T_j \cup \{ s \in \mathit{Sat}(\Phi) \mid \mathit{Post}(s) \cap T_j \neq \varnothing \}$
- Hence: $\{ s \in \mathit{Sat}(\Phi) \mid \mathit{Post}(s) \cap T_j \neq \varnothing \} \subseteq T_j$
  - hence, T_j satisfies (2), i.e., $(s \in \mathit{Sat}(\Phi)$ and $\mathit{Post}(s) \cap T_j \neq \varnothing) \Rightarrow s \in T_j$
  - further, $\mathit{Sat}(\Psi) = T_0 \subseteq T_j$ so, T_j satisfies (1), i.e. $\mathit{Sat}(\Psi) \subseteq T_j$
- As *Sat*$(\exists(\Phi \,\mathsf{U}\, \Psi))$ is the *smallest* set satisfying (1) and (2):
  - $\mathit{Sat}(\exists(\Phi \,\mathsf{U}\, \Psi)) \subseteq T_j$ and thus $\mathit{Sat}(\exists(\Phi \,\mathsf{U}\, \Psi)) = T_j$
- Hence: $T_0 \subsetneq T_1 \subsetneq T_2 \subsetneq \ldots \subsetneq T_j = T_{j+1} = \ldots = \mathit{Sat}(\exists(\Phi \,\mathsf{U}\, \Psi))$

Computing *Sat*$(\exists(\Phi \,\mathsf{U}\, \Psi))$ (3)

Input: finite transition system *TS* with state-set S and CTL-formula $\exists(\Phi \,\mathsf{U}\, \Psi)$
Output: *Sat*$(\exists(\Phi \,\mathsf{U}\, \Psi)) = \{ s \in S \mid s \models \exists(\Phi \,\mathsf{U}\, \Psi) \}$

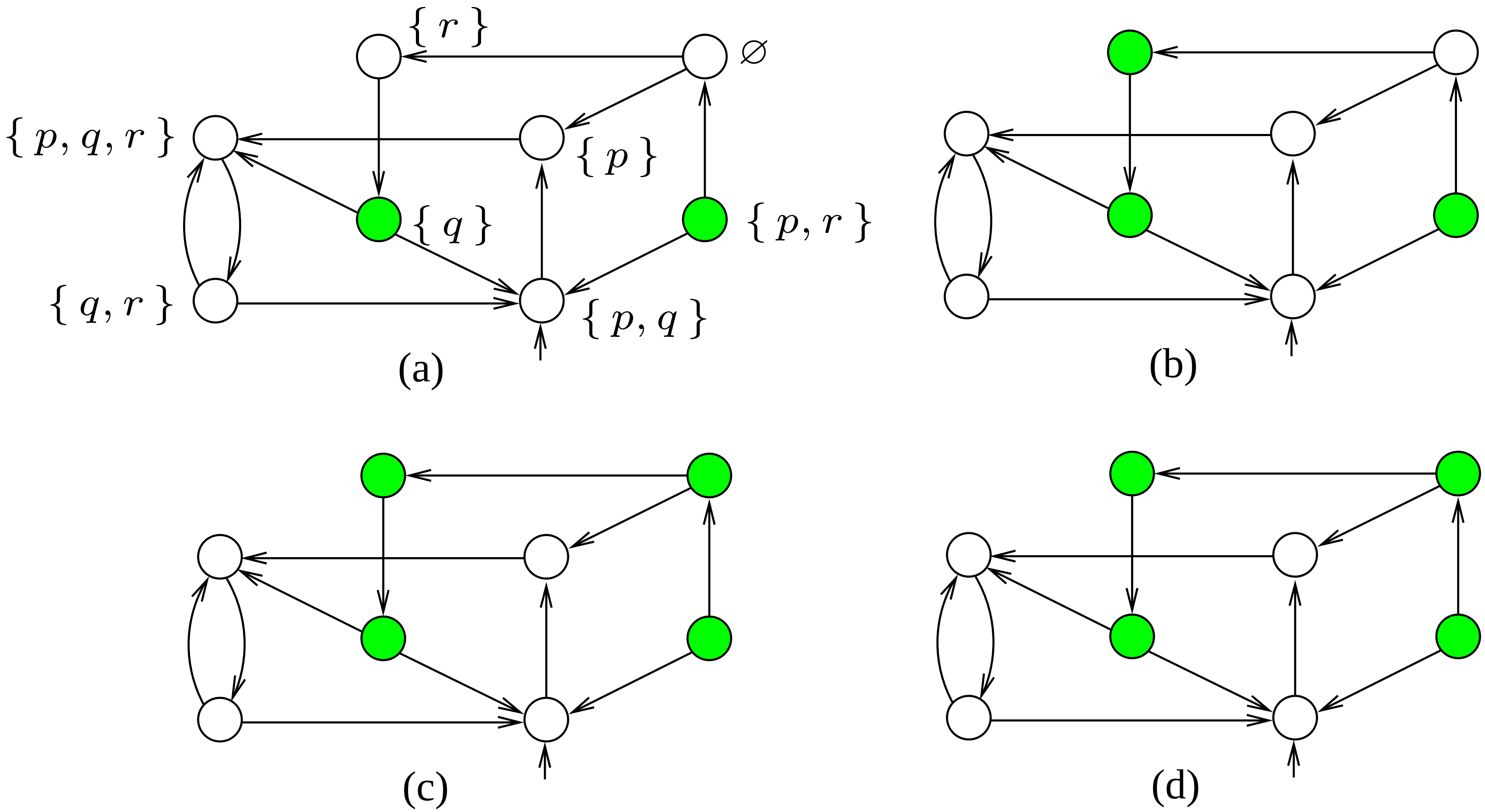
$E := \mathit{Sat}(\Psi)$; (* E administers the states s with $s \models \exists(\Phi \,\mathsf{U}\, \Psi)$ *)
$T := E$; (* T contains the already visited states s with $s \models \exists(\Phi \,\mathsf{U}\, \Psi)$ *)
while $E \neq \varnothing$ **do**
  let $s' \in E$;
  $E := E \setminus \{ s' \}$;
  for all $s \in \mathit{Pre}(s')$ **do**
    if $s \in \mathit{Sat}(\Phi) \setminus T$ **then** $E := E \cup \{ s \}; T := T \cup \{ s \}$; **endif**
  od
od
return T

Example



let's check the CTL-formula $\exists\Diamond\,((p = r) \wedge (p \neq q))$

The computation in snapshots



Computing *Sat*$(\exists\Box\,\Phi)$

Computing *Sat*$(\exists\Box\,\Phi)$

$E := S \setminus$ *Sat*(Φ); (* E contains any not visited s' with $s' \not\models \exists\Box\Phi$ *)

$T :=$ *Sat*(Φ); (* T contains any s for which $s \models \exists\Box\Phi$ has not yet been disproven *)

for all $s \in$ *Sat*(Φ) **do** $c[s] := |$ *Post*$(s)\,|$; **od** (* initialize array c *)

while $E \neq \varnothing$ **do**
(* loop invariant: $c[s] = |$ *Post*$(s) \cap (T \cup E)\,|$ *)
  let $s' \in E$; (* $s' \not\models \Phi$ *)
  $E := E \setminus \{ s' \}$; (* s' has been considered *)
  for all $s \in$ *Pre*(s') **do**
    if $s \in T$ **then**
      $c[s] := c[s] - 1$; (* update counter $c[s]$ for predecessor s of s' *)
      if $c[s] = 0$ **then**
        $T := T \setminus \{ s \}$; $E := E \cup \{ s \}$; (* s does not have any successor in T *)
      fi
    fi
  od
od
return T

Example

Alternative algorithm for $Sat(\exists\Box\,\Phi)$

1. Consider only state s if $s \models \Phi$, otherwise *eliminate* s
   - change *TS* into $TS[\Phi] = (S', Act, \rightarrow', I', AP, L')$ with $S' = Sat(\Phi)$,
   - $\rightarrow' = \rightarrow \cap (S' \times Act \times S')$, $I' = I \cap S'$, and $L'(s) = L(s)$ for $s \in S'$

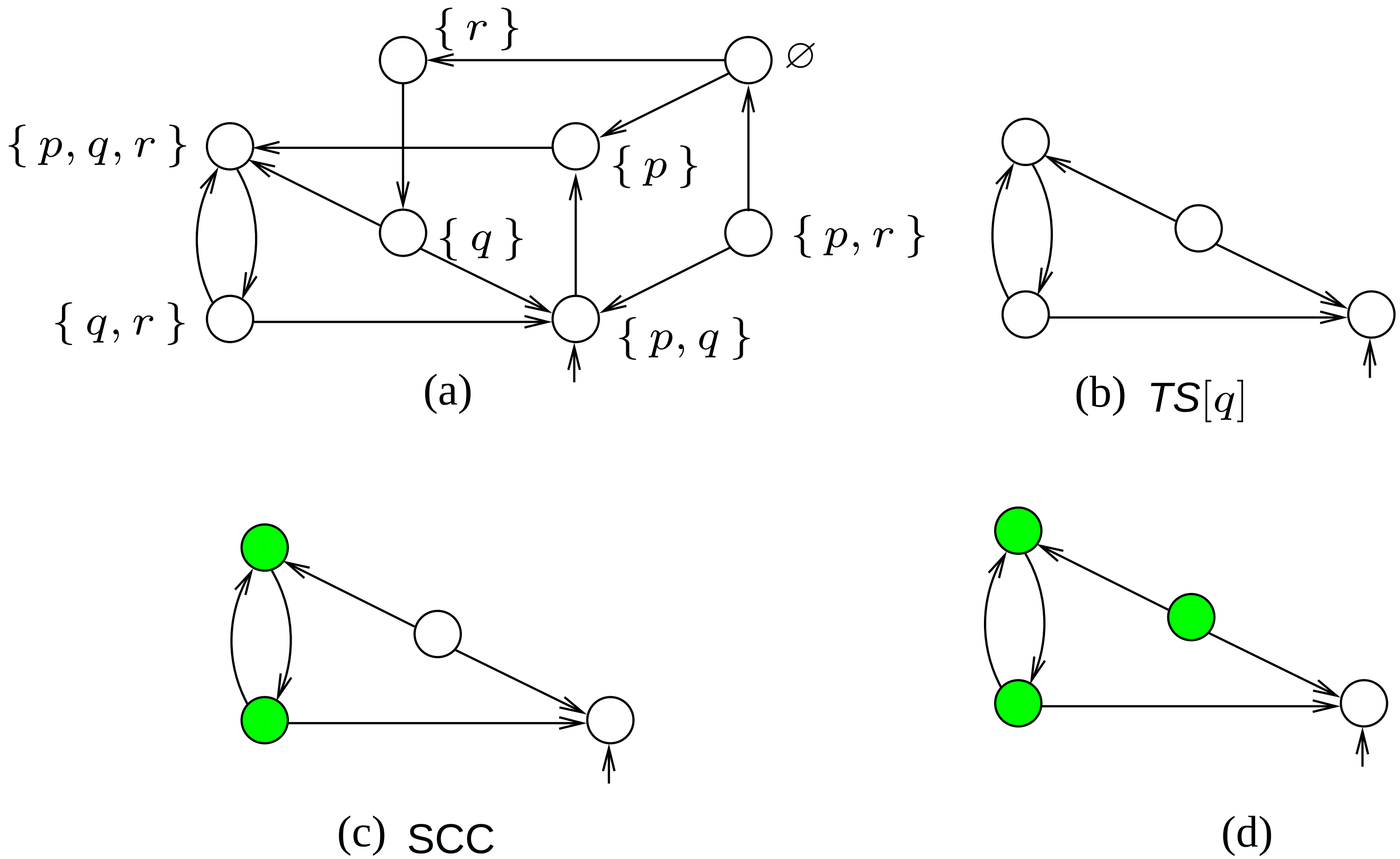
   $\Rightarrow$ all removed states will not satisfy $\exists\Box\,\Phi$, and thus can be safely removed

2. Determine all *non-trivial strongly connected components* in $TS[\Phi]$
   - non-trivial SCC = maximal, connected subgraph with at least one transition

   $\Rightarrow$ any state in such SCC satisfies $\exists\Box\,\Phi$

3. $s \models \exists\Box\,\Phi$ is equivalent to "some *SCC is reachable* from s"
   - this search can be done in a backward manner

Example



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$\Rightarrow$ Time complexity

Time complexity

For transition system *TS* with N states and K transitions,

and CTL formula Φ, the CTL model-checking problem *TS* $\models \Phi$

can be determined in time $\mathcal{O}(|\ \Phi\ |\cdot(N+M))$

this applies to both algorithms for $\exists\Box\ \Phi$

Model-checking LTL versus CTL

- Let *TS* be a transition system with N states and M transitions
- Model-checking LTL-formula Φ has time-complexity $\mathcal{O}((N+M) \cdot 2^{|\,\Phi\,|})$
  - linear in the state space of the system model
  - exponential in the length of the formula
- Model-checking CTL-formula Φ has time-complexity $\mathcal{O}((N+M) \cdot |\,\Phi\,|)$
  - linear in the state space of the system model and the formula
- Is model-checking CTL more efficient?

Model-checking LTL versus CTL

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- Model-checking LTL-formula Φ has time-complexity $\mathcal{O}((N+M) \cdot 2^{|\,\Phi\,|})$
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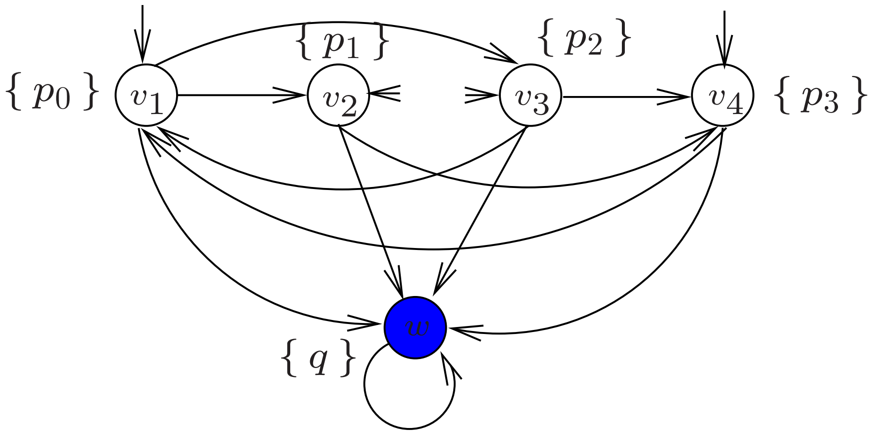
- Model-checking CTL-formula Φ has time-complexity $\mathcal{O}((N+M) \cdot |\,\Phi\,|)$
  - linear in the state space of the system model and the formula
- Is model-checking CTL more efficient? No!

Hamiltonian path problem (1)

$\Rightarrow$ LTL-formulae can be *exponentially shorter* than their CTL-equivalent



- Existence of Hamiltonian path in LTL: $\bigwedge_i \left(\Diamond p_i \;\wedge\; \Box (p_i \rightarrow \bigcirc \Box \neg p_i) \right)$
- In CTL, all possible (= 4!) routes need to be encoded