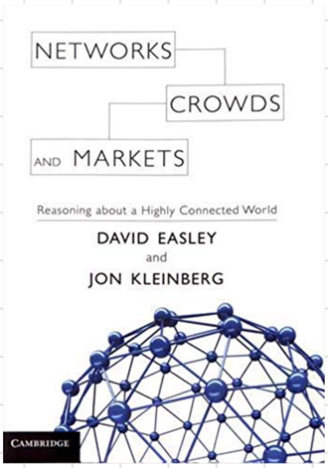
Lecture 15

Network Science

Analysis of Rich-Get-Richer Processes

Today's topics

- Analysis of Rich-Get-Richer processes



Chapter 18

"Power Laws and Rich-Get-Richer Phenomena"

Section 18.7

Analysis of Rich-Get-Richer Processes

$f(k)$: fraction of nodes with k (in) degree

goal : $f(k) \cong k^{-c}$

why this happens with a RGR model?

Which is the "meaning" of "c"?

RGR model:

1) nodes are created in order: $1, 2, \dots, N$

2) a new node J is created.

   2a) with prob. p, J will be connected to i uniformly at random

   2b) with prob. $1-p$, J will be connected to l, with a prob. proportional to l's current degree.

   3c) repeat the process from step 1
   (for the sake of simplicity only one link is created along with the new node)

$x_j(t)$ = random variable

= # of links to J at a time step t

·) $x_j(j) = 0$

··) $$x_j(t+1) = x_j(t) + \boxed{\frac{p}{t} + \frac{(1-p)\,x_j(t)}{t}}$$

expected change in $x_j(t)$

"deterministic argument"

$x_j(t)$: continuous function

$q = 1 - p$

$$\frac{dx_j}{dt} = \frac{p + q x_j}{t}$$

$$\Rightarrow \quad \frac{1}{p + q x_j} \cdot \frac{dx_j}{dt}\, dt = \frac{1}{t}\, dt$$

integrate on both sides

$$\int \frac{1}{p + q x_j} \cdot dx_j = \int \frac{1}{t}\, dt$$

$$q\left(\frac{\ln(p + q x_j)}{q} + C_1\right) = q\left(\ln t + C_2\right)$$

$$\ln(p + q x_j) = q \ln t + c$$

let $A = e^c$

exponentiate both sides

$$p + q x_j = A t^q$$

$$\cdot \quad x_j(t) = \frac{1}{q}\left(A t^q - p\right) \quad ★$$

recall initial condition $x_j(j) = 0$

$$0 = x_j(j) = \frac{1}{q}\left(A j^q - p\right)$$

$$A j^q - p = 0$$

$$x_j(j) = \frac{1}{q}\left(A j^q - p\right) = 0$$

$$A j^q - p = 0$$

$$\boxed{A = \frac{p}{j^q}}$$

substitute to ★

$$x_j(t) = \frac{1}{q}\left(\frac{p}{j^q} t^q - p\right) = \frac{p}{q}\left[\left(\frac{t}{j}\right)^q - 1\right]$$

$$x_j(t) = \frac{p}{q}\left[\left(\frac{t}{j}\right)^q - 1\right]$$

it is a closed form expression for how each x_j grows over time

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for a given value of k, and a time t, what fraction of all functions x_j satisfies $x_j(t) \geq k$?

$$x_j(t) = \frac{p}{q}\left[\left(\frac{t}{j}\right)^q - 1\right] \geq k$$

$$\Rightarrow j \leq t\left[\frac{q}{p}\cdot k + 1\right]^{-\frac{1}{q}}$$

out of all the functions $x_1, x_2, \dots, x_t$ at time t, the fraction of values j that satisfies this is:

$$\Rightarrow F(k) = \frac{1}{\cancel{t}}\cancel{t}\underbrace{\left[\frac{q}{p}k + 1\right]}_{\text{proportional to } k}^{-\frac{1}{q}}$$

neg exp.

F(k) : fraction of nodes with at least degree k

f(k) : fraction of nodes with exactly degree k.

Hence, we take the derivative:

$$-\frac{dF}{dk} = \frac{1}{\cancel{q}} \cdot \frac{\cancel{q}}{p}\left(\frac{q}{p}k + 1\right)^{-1-\frac{1}{q}}$$

$$f(k) \simeq \underbrace{\frac{1}{p}\left(\frac{q}{p}k + 1\right)}_{}{}^{\underbrace{-\left(1+\frac{1}{q}\right)}_{}}$$

$$c = 1 + \frac{1}{q} = 1 + \frac{1}{1-p}$$

this is a power law $\square$

Which is the meaning of c?

$$c = 1 + \frac{1}{1-p}$$

$$\lim_{p \to 0} \left(1 + \frac{1}{1-p}\right) = 2$$

.) the growths is mainly governed by preferential attachment ⇒

..) the power law exponent decreases toward 2, allowing to very large degrees to arise.

real networks : $2 \leq c < 3$

"scale-free regime"

$$\lim_{p \to 1} \left(1 + \frac{1}{1-p}\right) = \underline{\infty}$$

.) when $p \to 1$: random selection

..) RGR effect is not dominant

...) very large degrees are rare!

Take home message

i) We proved that a rich-get-richer process leads to the formation of a scale-free network

ii) the exponent of the power law provides a "measure" of the impact of random connection vs preferential attachment

iii) this model still does not explain the co-existence in real networks of short distances and high clustering coefficient.